

ALL COUNTABLE GROUPS HAVE CUBIC PRESENTATIONS

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ABSTRACT. Let G be a group with presentation such that each generator occurs at most countably many times in the set of relations. Then for all $n \geq 3$, G is n -ic. In particular, for all $n \geq 3$, countable groups have n -ic presentations.

Neuwirth [1] calls a group n -ic if it admits a presentation in which each generator appears exactly n times in the set of relations; such a presentation is also called n -ic. He goes on to show that if G is the fundamental group of a closed n -manifold then $G * F$ is n -ic for some free F . Hoare (unpublished; see [2, p. 146]) has shown for $n = 3$ that G itself is 3-ic (cubic), and Montesinos [3] extends this result to 3-manifolds with boundary. Here we show that any group that admits a presentation in which each generator occurs at most countably many times in the set of relations is n -ic.

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We shall employ the following conventions and notation:

Let X be a set and F the free group on X . Let R be a set of (reduced) words on X and $\langle\langle R \rangle\rangle_F$ the normal closure of R in F . Then we say $\langle X | R \rangle$ is a presentation for a group G if $G \cong F / \langle\langle R \rangle\rangle_F$. In this case we say G is generated by the set X with relations R . Now suppose there exist $r_1, r_2 \in R$ such that $wr_1w^{-1} = r_2^{\pm 1}$, in F , for some $w \in F$. Then $\langle\langle R \rangle\rangle_F = \langle\langle R - \{r_2\} \rangle\rangle_F$, so we may assume that no relation is conjugate to any other relation or its inverse without changing the group. Throughout this paper, 'presentation' means (cyclicly reduced) group presentation with no relation conjugate to any other relation or its inverse.

Notation. Given a presentation $\langle X | R \rangle$ with $x \in X$ and $S \subseteq R$, let $\#(x, S) =$ the number of times x occurs in the subset S of relations.

We may now state the main result.

THEOREM. *If G is a group with presentation such that each generator appears at most countably many times in the set of relations, then G is cubic.*

COROLLARY. *If G is cubic then for all $n \geq 3$: G is n -ic.*

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The proof is based on this rather simple idea: Suppose a generator x appears n times in R , where n is finite but larger than three. We modify the presentation in the following manner. First remove x from X and adjoin the formal symbols x_1, \dots, x_n . Next we remove the relations in which x appears from R and replace them with exactly the same relations except that the i th occurrence of x will be replaced by x_i for each i . Finally we adjoin the set of relations $\{x_1x_2^{-1}, x_2x_3^{-1}, \dots, x_{n-1}x_n^{-1}, x_nx_1^{-1}\}$ to R . This serves two purposes: it equates all the x_i 's so that we still present G ; and it insures that each x_i occurs exactly thrice. We apply a similar series of modifications for each offending generator to obtain the desired presentation.

EXAMPLE. From $\langle x, y | x^2y^2, xyx \rangle$ we get

$$\langle x_1, x_2, x_3, x_4, y | x_1x_2y^2, x_3yx_4, x_1x_2^{-1}, x_2x_3^{-1}, x_3x_4^{-1}, x_4x_1^{-1} \rangle.$$

PROOF OF THEOREM. Let $\langle X | R \rangle$ be a presentation for a group G satisfying the hypothesis. For $Y \subseteq X$ we define $R_Y = R$ with the k th occurrence of each $x \in Y$ replaced by x_k for each k . Note that $\#(x, R) = \#(x, R_Y)$ if $x \notin Y$, so we may define a set of generators

$$T_Y = \bigcup_{x \in Y} \{x_1, x_2, \dots, x_{\#(x, R_{(Y-(x))})}\}.$$

Finally, for each $x \in X$, if we could adjoin a set S_x of relations among the x_k 's which equate them all and in which each x_k appears exactly twice, then

$$\left\langle \bigcup_{x \in X} T_x | R_X \cup \left(\bigcup_{x \in X} S_x \right) \right\rangle$$

would be a cubic presentation for G since $\#(x_k, R_X) = 1$ and $\#(x_k, \bigcup_{x \in X} S_x) = \#(x_k, S_x) = 2$ for each $x \in X$. Unfortunately, we cannot find such S_x 's for $\#(x, R) < 3$; in this case we must adjoin a few 'dummy generators' T'_x and include in S_x relations amongst elements of T'_x and the x_k 's which equate them all, being careful to choose T'_x and S_x so that $\#(y, S_x) = 3$ for all $y \in T'_x$ and $\#(y, S_x) = 2$ for all $y \in T_x$. Then

$$\left\langle T_X \cup \left(\bigcup_{x \in X} T'_x \right) | R_X \cup \left(\bigcup_{x \in X} S_x \right) \right\rangle$$

will be our cubic presentation. There are five possible cases:

Case 0. $\#(x, R) = 0$.

$$\begin{aligned} T'_x &= \{y_1, \dots, y_4\}, & T_x &= \emptyset, \\ S_x &= \{y_1y_2^{-1}, y_2y_3^{-1}, y_3y_1^{-1}, y_1y_4^{-1}, y_2y_4^{-1}, y_3y_4^{-1}\}, \\ \forall y \in T'_x, & \#(y, S_x) = 3. \end{aligned}$$

Case 1. $\#(x, R) = 1$.

$$\begin{aligned} T'_x &= \{y_1, y_2, z_1, z_2\}, & T_x &= \{x_1\}, \\ S_x &= \{x_1z_1^{-1}, x_1z_2^{-1}, z_1y_1^{-1}, z_1y_2^{-1}, z_2y_1^{-1}, z_2y_2^{-1}, y_1y_2^{-1}\}, \\ \#(x_1, S_x) &= 2; \quad \forall y \in T'_x, \#(y, S_x) = 3. \end{aligned}$$

Case 2. $\#(x, R) = 2$.

$$\begin{aligned} T'_x &= \{y_1, y_2\}, & T_x &= \{x_1, x_2\}, \\ S_x &= \{x_1y_1^{-1}, x_1y_2^{-1}, x_2y_1^{-1}, x_2y_2^{-1}, y_1y_2^{-1}\}, \\ \forall x_k \in T_x, \#(x_k, S_x) &= 2; & \forall y_k \in T'_x, \#(y_k, S_x) &= 3. \end{aligned}$$

Case 3. $3 \leq \#(x, R) < \infty$.

$$\begin{aligned} T'_x &= \emptyset, & T_x &= \{x_1, \dots, x_n\} \text{ where } n = \#(x, R), \\ S_x &= \{x_1x_2^{-1}, x_2x_3^{-1}, \dots, x_{n-1}x_n^{-1}, x_nx_1^{-1}\}, \\ \forall x_k \in T_x, \#(x_k, S_x) &= 2. \end{aligned}$$

Case 4. $\#(x, R)$ is countably infinite.

$$\begin{aligned} T'_x &= \emptyset, & T_x &= \{x_1, x_2, \dots\}, \\ S_x &= \{\dots, x_{2i}x_{2i-2}^{-1}, \dots, x_6x_4^{-1}, x_4x_2^{-1}, x_2x_1^{-1}, \\ & \qquad \qquad \qquad x_1x_3^{-1}, x_3x_5^{-1}, \dots, x_{2i+1}x_{2i+3}^{-1}, \dots\}, \\ \forall x_k \in T_x, \#(x_k, S_x) &= 2. \end{aligned}$$

In each case note that no two elements of S_x are conjugate (in the free group) since each relation has length two, so conjugation either increases length or reverses the order of the elements of each word. Also, $r_1, r_2 \in R_x$ implies r_1 and r_2 have no letters in common, implying that r_1 is not conjugate to $r_2^{\pm 1}$ (it was for this reason that we included $\#(x, R) = 3$ in the third case). Finally, any conjugate of a word in S_x (or its inverse) contains $\dots a_i a_j^{-1} \dots$, where $a_i, a_j \in T_x \cup T'_x$. But R_x cannot contain any word with this configuration, since this implies R contains xx^{-1} , contrary to hypothesis.

PROOF OF COROLLARY. Suppose G has an $(n - 1)$ -ic presentation $\langle X|R \rangle$. Let \bar{X} be another copy of X whose elements are distinguished from those of X by writing a bar above them. Then $\langle \bar{X}|\bar{R} \rangle$ is another $(n - 1)$ -ic presentation for G . Now we equate corresponding elements of X and \bar{X} by adjoining the set of relations $S = \{x\bar{x}^{-1} | x \in X\}$. Note that $\langle X \cup \bar{X} | R \cup \bar{R} \cup S \rangle$ is a presentation for G , since in the free group on $X \cup \bar{X}$, we have

$$\begin{aligned} \langle \langle R \rangle \rangle \cap \langle \langle S \rangle \rangle &= \langle \langle \bar{R} \rangle \rangle \cap \langle \langle S \rangle \rangle = \langle \langle R \rangle \rangle \cap \langle \langle \bar{R} \rangle \rangle \\ &= \text{the empty (identity) word.} \end{aligned}$$

Finally, for all $x \in X$,

$\#(x, R \cup \bar{R} \cup S) = \#(x, R) + \#(x, \bar{R}) + \#(x, S) = (n - 1) + 0 + 1 = n$, and similarly for each \bar{x} , so in fact we get an n -ic presentation for G . Using the above theorem, we complete the proof by induction.

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